

Absorption of a Quantum in a D1/D5 System

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Abstract

In *Nucl. Phys. B* **615**, 285 (2001) [arXiv:hep-th/0107113], the wave equation for a minimally coupled scalar was studied in the geometry of a D1-D5 system with non-zero angular momentum. The probability for a quantum to enter the throat was computed by taking small a parameter γ which is associated with the value of the angular momentum. In the leading order in γ , the result was found to agree with the dual CFT result. In this note, we report on an observation that there are corrections of higher order in γ . Our results should be useful for determining the higher order correction terms that the dual CFT needs in order to incorporate the presence of the ‘capped’ geometry.

A D1/D5 brane system has been a valuable asset for studying black hole physics leading to many fruitful results in the calculations of entropy and radiation rate. For this system, the microscopic consideration yielded exactly the same value as the Bekenstein entropy [1, 2]. Similarly precise agreement was found between its low energy radiation rate and the Hawking radiation of the corresponding black hole [3, 4].

It has also served as a stage where one tackles the black hole information puzzle. A recent attempt to resolve the information puzzle can be found in [5, 6]. The system that was considered in these works is not the conventional D1/D5 brane which has an infinite throat, but rather the one with its throat ‘capped’ off at a finite distance in the conical singularity. The way the throat caps off maps to a dual CFT microstate. The mapping was established in [6, 8] for special family of metrics providing another test for AdS/CFT correspondence.

In [9], the wave equation for a minimally coupled scalar was studied in the geometry of these special family of metrics. The truncated throat was called a ‘tube’. Most of the low energy quanta that are sent toward the throat from spatial infinity get reflected back to the infinity when they arrive at the start of the throat. But there is a small probability for them to enter the throat and travel until they reach the end of the throat. Once reaching the end, they get reflected back to the start of the throat where they have the same small probability to exit from the tube.

The probability for a quantum to enter the throat was obtained by the same authors. They also computed the ‘time delay’ for two low frequency quanta that emerge from the tube successively. However for both of these computations a parameter γ which is associated with the value of the angular momentum was taken small. In the leading order in γ , agreements were found with the corresponding CFT computations. Since the presence of γ itself is tied with the fact that the geometry is capped, it is worth re-analyzing the problem in the sub-leading order in γ in order to see whether corrections occur.

This is, in fact, what we will do in this note: we keep γ arbitrary, and re-compute the time delay and the probability. Also in [9] the computation was done with null momentum along the compact direction of D1 brane wrapping. We will lift this condition

too since it is not difficult to do so. As we will see below, the time delay turns out to be the same. However, the probability gets corrected in γ (and in λ as well). In other words, for a generic value of the parameter γ the gravity calculation of the probability gives a different result than the one obtained from the CFT analysis although the former gets reduced to the latter in the small γ limit. The discrepancy could probably be accounted by putting correction terms, which arise from a DBI action, in the CFT to incorporate the presence of the ‘capped’ geometry. Our results should be useful to determine them.

We begin by reviewing [9]. Let R be the radius of the circle that is common to the D1 and D5 branes, and n_1, n_5 the number of D1, D5 branes respectively. The metric for the rotating D1/D5 brane system [10, 7] is:

$$ds^2 = -\frac{1}{h}(dt^2 - dy^2) + hf \left(d\theta^2 + \frac{dr^2}{r^2 + a^2} \right) - \frac{2a\sqrt{Q_1 Q_5}}{hf} (\cos^2 \theta dy d\psi + \sin^2 \theta dt d\phi) \\ + h \left[\left(r^2 + \frac{a^2 Q_1 Q_5 \cos^2 \theta}{h^2 f^2} \right) \cos^2 \theta d\psi^2 + \left(r^2 + a^2 - \frac{a^2 Q_1 Q_5 \sin^2 \theta}{h^2 f^2} \right) \sin^2 \theta d\phi^2 \right], \quad (1)$$

where

$$f = r^2 + a^2 \cos^2 \theta, \quad h = \left[\left(1 + \frac{Q_1}{f} \right) \left(1 + \frac{Q_5}{f} \right) \right]^{1/2} \quad (2)$$

The parameter a has a physical meaning that the throat ends at $r \sim a$. It was re-written in [9] in terms of a dimensionless parameter $\gamma = \frac{2J}{n_1 n_5}$ where J represents the angular momentum:

$$a = \frac{\sqrt{Q_1 Q_5}}{R} \gamma \quad (3)$$

We take $R \gg (Q_1 Q_5)^{1/4}$. Consider the wave equation for a minimally coupled scalar in this background. For a massless scalar, it is

$$\square \Phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) = 0. \quad (4)$$

Writing the solution in the following form:

$$\Phi(t, r, \theta, \phi, \psi, y) = \exp(-i\tilde{\omega}t + im\phi + in\psi + i\tilde{\lambda}y) \tilde{\Phi}(r, \theta). \quad (5)$$

with

$$\tilde{\Phi}(r, \theta) = H(r)\Theta(\theta), \quad (6)$$

equation (4) turns into a set of two equations, one for the radial part and the other for the angular part. The existence of separation of variables for such a wave equation was shown in [11]. Denoting the eigenvalue of the angular equation by Λ , its approximate value was determined in [9]:

$$\Lambda = l(l+2) + O((a\tilde{\omega})^2) + O((a\tilde{\lambda})^2) \quad (7)$$

where l is a nonnegative integer that specifies the spherical harmonic of the angular solution. By introducing a new coordinate

$$x = \frac{r^2 R^2}{Q_1 Q_5} \quad (8)$$

and dimensionless parameters $\omega = R\tilde{\omega}$, $\lambda = R\tilde{\lambda}$ the radial equation can be put into the following form,

$$4 \frac{d}{dx} \left(x(x + \gamma^2) \frac{dH}{dx} \right) + \left\{ (\omega^2 - \lambda^2) \left[\frac{Q_1 Q_5}{R^4} x + \frac{Q_1 + Q_5}{R^2} \right] + \frac{(\omega - m\gamma)^2}{x + \gamma^2} - \frac{(\lambda + n\gamma)^2}{x} \right\} H - \Lambda H = 0 \quad (9)$$

A solution was found in [9] imposing $\lambda = 0$ and taking $\gamma \rightarrow 0$. Here we relax these constraints.

As in [9], we solve (9) in two asymptotic regions and match the solutions obtained in each region. First consider the large- x region. In this region we can borrow the corresponding steps of [9] with minimal modifications. For self-containedness of this note, we enclose some details: (9) simplifies to

$$4x^2 H'' + 8x H' + (\omega^2 - \lambda^2) \left\{ \frac{Q_1 Q_5}{R^4} x + \frac{Q_1 + Q_5}{R^2} \right\} H - \Lambda H = 0 \quad (10)$$

which is the same as the corresponding equation in [9] other than the shift, $w^2 \rightarrow \omega^2 - \lambda^2$. The general solution is a linear combination of Bessel functions, but now with a shifted argument,

$$H_{out}(x) = \frac{1}{\sqrt{x}} \left[C_1 J_\nu \left(\sqrt{\frac{Q_1 Q_5 (\omega^2 - \lambda^2) x}{R^4}} \right) + C_2 J_{-\nu} \left(\sqrt{\frac{Q_1 Q_5 (\omega^2 - \lambda^2) x}{R^4}} \right) \right], \quad (11)$$

where

$$\nu = \left(1 + \Lambda - (\omega^2 - \lambda^2) \frac{Q_1 + Q_5}{R^2}\right)^{1/2} \equiv l + 1 + \epsilon \quad (12)$$

and

$$\epsilon \approx -\frac{(\omega^2 - \lambda^2)}{l + 1} \frac{Q_1 + Q_5}{2R^2} \quad (13)$$

Expanding the Bessel functions in the leading order in the region where

$$\frac{Q_1 Q_5 (\omega^2 - \lambda^2) x}{R^4} \ll 1 \quad (14)$$

one gets

$$H_{out}(x) \approx \frac{1}{\sqrt{x}} \left(\frac{Q_1 Q_5 (\omega^2 - \lambda^2) x}{4R^4} \right)^{-\frac{l+1}{2}} \left[-(-1)^l C_2 \epsilon l! + \frac{C_1 - (-1)^l C_2}{(l+1)!} \left(\frac{Q_1 Q_5 (\omega^2 - \lambda^2) x}{4R^4} \right)^{l+1} \right]. \quad (15)$$

Now let's turn to the inner region. The inner region is defined by

$$x \ll \frac{(Q_1 + Q_5) R^2}{Q_1 Q_5} \quad (16)$$

Then (9) can be approximated to:

$$4 \frac{d}{dx} \left(x(x + \gamma^2) \frac{dH}{dx} \right) + \left\{ (\omega^2 - \lambda^2) \left[\frac{Q_1 + Q_5}{R^2} \right] + \frac{(\omega - m\gamma)^2}{x + \gamma^2} - \frac{(\lambda + n\gamma)^2}{x} \right\} H = \Lambda H \quad (17)$$

By taking

$$H_{in}(x) = x^\alpha (\gamma^2 + x)^\beta G(x), \quad (18)$$

where

$$\alpha = \frac{1}{2} \left(n + \frac{\lambda}{\gamma} \right), \quad \beta = \frac{w - \gamma m}{2\gamma} \quad (19)$$

equation (17) is converted to the hypergeometric differential equation

$$4x(x + \gamma^2) G'' + 4 [2x(\alpha + \beta + 1) + \gamma^2(1 + 2\alpha)] G' + \left\{ 4(\alpha + \beta)(\alpha + \beta + 1) + (\omega^2 - \lambda^2) \frac{Q_1 + Q_5}{R^2} - \Lambda \right\} G = 0 \quad (20)$$

The solution that is regular at $x = 0$ is

$$G(x) = F\left(p, q; 1 + 2\alpha; -\frac{x}{\gamma^2}\right) \quad (21)$$

with

$$p = \frac{1}{2} + \alpha + \beta + \frac{1}{2}\sqrt{1 + \Lambda - (\omega^2 - \lambda^2)\frac{Q_1 + Q_5}{R^2}} \quad (22)$$

$$q = \frac{1}{2} + \alpha + \beta - \frac{1}{2}\sqrt{1 + \Lambda - (\omega^2 - \lambda^2)\frac{Q_1 + Q_5}{R^2}} \quad (23)$$

For further manipulation, it is convenient to re-express $G(x)$ using one of the hypergeometric identities,

$$\begin{aligned} G(x) &= \frac{\Gamma(1 + 2\alpha)\Gamma(-\nu')}{\Gamma(\frac{1}{2} + \alpha + \beta - \frac{\nu'}{2})\Gamma(\frac{1}{2} + \alpha - \beta - \frac{\nu'}{2})} \left(\frac{x}{\gamma^2}\right)^{-p} F\left(p, p - 2\alpha; \nu' + 1; -\frac{\gamma^2}{x}\right) \\ &+ \frac{\Gamma(1 + 2\alpha)\Gamma(\nu')}{\Gamma(\frac{1}{2} + \alpha + \beta + \frac{\nu'}{2})\Gamma(\frac{1}{2} + \alpha - \beta + \frac{\nu'}{2})} \left(\frac{x}{\gamma^2}\right)^{-q} F\left(q, q - 2\alpha; -\nu' + 1; -\frac{\gamma^2}{x}\right) \end{aligned} \quad (24)$$

In [9], the leading behavior of $G(x)$ was obtained by first approximating the hypergeometric functions to the Bessel functions. Here we do not take these intermediate steps but rather directly deal with the hypergeometric functions.

Noting that the hypergeometric function has the following series expansion

$$F(p, p - 2\alpha; c; -z) = \sum_{n=0}^{\infty} \frac{\Gamma(p + n)}{\Gamma(p)} \frac{\Gamma(p - 2\alpha + n)}{\Gamma(p - 2\alpha)} \frac{\Gamma(c)}{\Gamma(c + n)} \frac{(-z)^n}{n!} \quad (25)$$

one can approximate (24):

$$\begin{aligned} G(x) &\simeq \frac{\Gamma(1 + 2\alpha)\Gamma(-\nu')}{\Gamma(\frac{1}{2} + \alpha + \beta - \frac{\nu'}{2})\Gamma(\frac{1}{2} + \alpha - \beta - \frac{\nu'}{2})} \left(\frac{x}{\gamma^2}\right)^{-p} \\ &+ \frac{\Gamma(1 + 2\alpha)\Gamma(\nu')}{\Gamma(\frac{1}{2} + \alpha + \beta + \frac{\nu'}{2})\Gamma(\frac{1}{2} + \alpha - \beta + \frac{\nu'}{2})} \left(\frac{x}{\gamma^2}\right)^{-q} (1 + Y) \end{aligned} \quad (26)$$

where¹

$$\begin{aligned} Y &\equiv \xi \frac{\Gamma(-l - \epsilon')}{\Gamma(1 - \epsilon')} \frac{(-z)^{l+1}}{(l+1)!} \\ &\simeq \xi \frac{1}{\epsilon' l! (l+1)!} \frac{z^{l+1}}{\Gamma(1 - \epsilon')} \end{aligned} \quad (27)$$

¹In [9], another symbol ϵ' was introduced for the inner region: it is the same as ϵ . We simply adopt their notation.

with $z \equiv \frac{\gamma^2}{x}$ and

$$\begin{aligned}\xi &\equiv \frac{\Gamma(q+l+1)}{\Gamma(q)} \frac{\Gamma(q-2\alpha+l+1)}{\Gamma(q-2\alpha)} \\ &= (q+l) \cdots q (q-2\alpha+l) \cdots (q-2\alpha)\end{aligned}\tag{28}$$

Assuming $x \gg \gamma^2$

$$\begin{aligned}H_{in} &\simeq x^{\alpha+\beta} G(x) \\ &\simeq \frac{\Gamma(1+2\alpha)\Gamma(-\nu')}{\Gamma(\frac{1}{2}+\alpha+\beta-\frac{\nu'}{2})\Gamma(\frac{1}{2}+\alpha-\beta-\frac{\nu'}{2})} \left(\frac{x}{\gamma^2}\right)^{\alpha+\beta-p} \gamma^{2(\alpha+\beta)} \\ &\quad + \frac{\Gamma(1+2\alpha)\Gamma(\nu')}{\Gamma(\frac{1}{2}+\alpha+\beta+\frac{\nu'}{2})\Gamma(\frac{1}{2}+\alpha-\beta+\frac{\nu'}{2})} \left(\frac{x}{\gamma^2}\right)^{\alpha+\beta-q} \gamma^{2(\alpha+\beta)} (1+Y) \\ &\simeq \frac{\gamma^{1+2(\alpha+\beta)}}{\sqrt{x}} \frac{\Gamma(1+2\alpha)}{\Gamma(\frac{1}{2}+\alpha+\beta+\frac{\nu'}{2})\Gamma(\frac{1}{2}+\alpha-\beta+\frac{\nu'}{2})} \\ &\quad \left[\frac{\Gamma(\frac{1}{2}+\alpha+\beta+\frac{\nu'}{2})\Gamma(\frac{1}{2}+\alpha-\beta+\frac{\nu'}{2})}{\Gamma(\frac{1}{2}+\alpha+\beta-\frac{\nu'}{2})\Gamma(\frac{1}{2}+\alpha-\beta-\frac{\nu'}{2})} \Gamma(-\nu') \left(\frac{x}{\gamma^2}\right)^{-\frac{1}{2}(l+1)} \right. \\ &\quad \left. + \frac{\Gamma(\nu')}{\Gamma(1-\epsilon')} \xi \frac{1}{\epsilon' l!(l+1)!} \left(\frac{x}{\gamma^2}\right)^{-\frac{1}{2}(l+1)} + \Gamma(\nu') \left(\frac{x}{\gamma^2}\right)^{\frac{1}{2}(l+1)} \right]\end{aligned}\tag{29}$$

Imagine a low frequency quantum that is sent toward $r = 0$ from spatial infinity. As shown in [9], it is likely to be reflected at the start of the throat. But there is a small probability, P , for the quantum to enter the throat. Once it enters the throat, it travels to the end of the throat. There it gets reflected back to the start of the throat and could leave the throat toward spatial infinity with the same probability, P . Consider a quantity R , whose absolute square is the reflection coefficient. Its explicit expression is obtained in the first equation of (4.22) of [9], which we quote here,

$$R = e^{-i\pi\epsilon} - (1 - e^{-2i\pi\epsilon}) \frac{C_2(-1)^l}{C_1 - C_2(-1)^l e^{-i\pi\epsilon}}\tag{30}$$

Note that

$$\begin{aligned}
\frac{C_2(-1)^l}{C_1 - (-1)^l C_2 e^{-i\pi\epsilon}} &\simeq \frac{C_2(-1)^l}{C_1 - (-1)^l C_2 (1 - i\pi\epsilon)} \\
&\simeq \frac{C_2(-1)^l}{C_1 - (-1)^l C_2 + (-1)^l C_2 i\pi\epsilon} \\
&= \frac{\frac{C_2(-1)^l}{C_1 - (-1)^l C_2}}{1 + \frac{C_2(-1)^l}{C_1 - (-1)^l C_2} i\pi\epsilon}
\end{aligned} \tag{31}$$

As we will see below, $\frac{C_2(-1)^l}{C_1 - (-1)^l C_2} i\pi\epsilon$ is small, therefore we have

$$\frac{C_2(-1)^l}{C_1 - (-1)^l C_2 e^{-i\pi\epsilon}} \simeq \frac{C_2(-1)^l}{C_1 - (-1)^l C_2} \tag{32}$$

To obtain the matching conditions, compare the inner solution (29) with the outer solution (15). Comparison of the coefficients of the negative power of x gives

$$\begin{aligned}
&\left(\frac{Q_1 Q_5 (w^2 - \lambda^2)}{4R^4} \right)^{-\frac{l+1}{2}} (-1)^{l+1} C_2 \epsilon l! \\
&\simeq \gamma^{1+2(\alpha+\beta)} \frac{\Gamma(1+2\alpha)}{\Gamma(\frac{1}{2} + \alpha + \beta + \frac{\nu'}{2}) \Gamma(\frac{1}{2} + \alpha - \beta + \frac{\nu'}{2})} \\
&\left[\frac{\Gamma(\frac{1}{2} + \alpha + \beta + \frac{\nu'}{2}) \Gamma(\frac{1}{2} + \alpha - \beta + \frac{\nu'}{2})}{\Gamma(\frac{1}{2} + \alpha + \beta - \frac{\nu'}{2}) \Gamma(\frac{1}{2} + \alpha - \beta - \frac{\nu'}{2})} \Gamma(-\nu') \gamma^{l+1} \right. \\
&\quad \left. + \frac{\Gamma(\nu')}{\Gamma(1-\epsilon')} \xi \frac{1}{\epsilon' l! (l+1)!} \gamma^{l+1} \right]
\end{aligned} \tag{33}$$

Solving for C_2 , one gets

$$\begin{aligned}
C_2 &\simeq \left(\frac{Q_1 Q_5 (w^2 - \lambda^2)}{4R^4} \right)^{\frac{l+1}{2}} \frac{(-1)^{l+1} \gamma^{1+2(\alpha+\beta)+l+1} \Gamma(1+2\alpha)}{\Gamma(\frac{1}{2} + \alpha + \beta + \frac{\nu'}{2}) \Gamma(\frac{1}{2} + \alpha - \beta + \frac{\nu'}{2})} \\
&\frac{1}{\epsilon \epsilon' l! (l+1)!} \left[\frac{\Gamma(\frac{1}{2} + \alpha + \beta + \frac{\nu'}{2}) \Gamma(\frac{1}{2} + \alpha - \beta + \frac{\nu'}{2})}{\Gamma(\frac{1}{2} + \alpha + \beta - \frac{\nu'}{2}) \Gamma(\frac{1}{2} + \alpha - \beta - \frac{\nu'}{2})} (-1)^l + \frac{\Gamma(\nu')}{\Gamma(1-\epsilon')} \xi \frac{1}{l!} \right]
\end{aligned} \tag{34}$$

The other matching condition is

$$\begin{aligned}
&\left(\frac{Q_1 Q_5 (w^2 - \lambda^2)}{4R^4} \right)^{\frac{l+1}{2}} \frac{C_1 - (-1)^l C_2}{(l+1)!} \\
&\simeq \gamma^{1+2(\alpha+\beta)-l-1} \frac{\Gamma(1+2\alpha) \Gamma(\nu')}{\Gamma(\frac{1}{2} + \alpha + \beta + \frac{\nu'}{2}) \Gamma(\frac{1}{2} + \alpha - \beta + \frac{\nu'}{2})}
\end{aligned} \tag{35}$$

which gives

$$\begin{aligned} & \frac{1}{C_1 - (-1)^l C_2} \\ \simeq & \left(\frac{Q_1 Q_5 (w^2 - \lambda^2)}{4R^4} \right)^{\frac{l+1}{2}} \frac{\gamma^{l-2(\alpha+\beta)}}{(l+1)! \Gamma(\nu')} \frac{\Gamma(\frac{1}{2} + \alpha + \beta + \frac{\nu'}{2}) \Gamma(\frac{1}{2} + \alpha - \beta + \frac{\nu'}{2})}{\Gamma(1 + 2\alpha)} \end{aligned} \quad (36)$$

From this and (34), one gets

$$\begin{aligned} \frac{C_2}{C_1 - (-1)^l C_2} \simeq & \left(\frac{Q_1 Q_5 (w^2 - \lambda^2)}{4R^4} \right)^{l+1} (-1)^{l+1} \gamma^{2(l+1)} \frac{1}{\epsilon \epsilon' l! [(l+1)!]^2} \frac{1}{\Gamma(\nu')} \\ & \left[\frac{\Gamma(\frac{1}{2} + \alpha + \beta + \frac{\nu'}{2}) \Gamma(\frac{1}{2} + \alpha - \beta + \frac{\nu'}{2})}{\Gamma(\frac{1}{2} + \alpha + \beta - \frac{\nu'}{2}) \Gamma(\frac{1}{2} + \alpha - \beta - \frac{\nu'}{2})} (-1)^l + \frac{\Gamma(\nu')}{\Gamma(1 - \epsilon')} \xi \frac{1}{l!} \right] \end{aligned} \quad (37)$$

Noting that

$$\begin{aligned} \Gamma(\nu') &= \Gamma(l+1+\epsilon') \simeq l! \left(1 + \epsilon' \sum_{n=1}^l \frac{1}{n} \right) (1 - \gamma_0 \epsilon') \\ \Gamma(1+x) &\simeq 1 - \gamma_0 x \end{aligned} \quad (38)$$

where γ_0 is Euler's constant and using a formula derived in the appendix, (A.6), we get

$$\begin{aligned} \frac{C_2 (-1)^l}{C_1 - (-1)^l C_2} \simeq & \left(\frac{Q_1 Q_5 (w^2 - \lambda^2)}{4R^4} \right)^{l+1} \gamma^{2(l+1)} \frac{\xi_0}{\epsilon [l! (l+1)!]^2} \\ & \left[\frac{1}{2} \left(\frac{\Gamma'(1 + \alpha + \beta + \frac{l}{2})}{\Gamma(1 + \alpha + \beta + \frac{l}{2})} + \frac{\Gamma'(1 + \beta - \alpha + \frac{l}{2})}{\Gamma(1 + \beta - \alpha + \frac{l}{2})} \right. \right. \\ & \left. \left. + \frac{\Gamma'(\alpha + \beta - \frac{l}{2})}{\Gamma(\alpha + \beta - \frac{l}{2})} + \frac{\Gamma'(\beta - \alpha - \frac{l}{2})}{\Gamma(\beta - \alpha - \frac{l}{2})} \right) + \pi \frac{\cos \pi(\beta - \alpha + \frac{l}{2})}{\sin \pi(\beta - \alpha + \frac{l}{2})} \right. \\ & \left. - \sum_{n=1}^l \frac{1}{n} + 2\gamma_0 - \frac{\xi_1}{\xi_0} \right] \end{aligned} \quad (39)$$

where the prime on Γ represents a derivative with respect to the argument and $\xi \equiv \xi_0 + \epsilon' \xi_1 + \dots$. The first two coefficients, ξ_0 and ξ_1 , are derived in the appendix. They

are given by

$$\begin{aligned}
\xi_0 &= \frac{\Gamma(1 + \alpha + \beta + \frac{l}{2})}{\Gamma(\alpha + \beta - \frac{l}{2})} \frac{\Gamma(1 + \beta - \alpha + \frac{l}{2})}{\Gamma(\beta - \alpha - \frac{l}{2})} \\
\frac{\xi_1}{\xi_0} &= -\frac{1}{2} \frac{\Gamma'(1 + \alpha + \beta + \frac{l}{2})}{\Gamma(1 + \alpha + \beta + \frac{l}{2})} - \frac{1}{2} \frac{\Gamma'(1 + \beta - \alpha + \frac{l}{2})}{\Gamma(1 + \beta - \alpha + \frac{l}{2})} \\
&\quad + \frac{1}{2} \frac{\Gamma'(\alpha + \beta - \frac{l}{2})}{\Gamma(\alpha + \beta - \frac{l}{2})} + \frac{1}{2} \frac{\Gamma'(\beta - \alpha - \frac{l}{2})}{\Gamma(\beta - \alpha - \frac{l}{2})}
\end{aligned} \tag{40}$$

Substituting (40) into (39), one gets

$$\begin{aligned}
\frac{C_2(-1)^l}{C_1 - (-1)^l C_2} &\simeq \left(\frac{Q_1 Q_5 (w^2 - \lambda^2)}{4R^4} \right)^{l+1} \gamma^{2(l+1)} \frac{\xi_0}{\epsilon [l!(l+1)!]^2} \\
&\quad \left[\frac{\Gamma'(1 + \alpha + \beta + \frac{l}{2})}{\Gamma(1 + \alpha + \beta + \frac{l}{2})} + \frac{\Gamma'(1 + \beta - \alpha + \frac{l}{2})}{\Gamma(1 + \beta - \alpha + \frac{l}{2})} \right. \\
&\quad \left. + \pi \frac{\cos \pi(\beta - \alpha + \frac{l}{2})}{\sin \pi(\beta - \alpha + \frac{l}{2})} - \sum_{n=1}^l \frac{1}{n} + 2\gamma_0 \right]
\end{aligned} \tag{41}$$

This, combined with (32), allows one to re-write (30),

$$\begin{aligned}
R &= e^{-i\pi\epsilon} - 2\pi i \left(\frac{Q_1 Q_5 (w^2 - \lambda^2)}{4R^4} \right)^{l+1} \gamma^{2(l+1)} \frac{\xi_0}{[l!(l+1)!]^2} \\
&\quad \left[\frac{\Gamma'(1 + \alpha + \beta + \frac{l}{2})}{\Gamma(1 + \alpha + \beta + \frac{l}{2})} + \frac{\Gamma'(1 + \beta - \alpha + \frac{l}{2})}{\Gamma(1 + \beta - \alpha + \frac{l}{2})} \right. \\
&\quad \left. + \pi \frac{\cos \pi(\beta - \alpha + \frac{l}{2})}{\sin \pi(\beta - \alpha + \frac{l}{2})} - \sum_{n=1}^l \frac{1}{n} + 2\gamma_0 \right]
\end{aligned} \tag{42}$$

Depending on the number of trips that a quantum makes inside the tube, each of them has a different phase. To discover the phase, we rewrite the $\frac{\cos}{\sin}$ -term as

$$\begin{aligned}
\frac{\cos \pi(\beta - \alpha + \frac{l}{2})}{\sin \pi(\beta - \alpha + \frac{l}{2})} &= -i \frac{1 + e^{2\pi(\beta - \alpha + l/2)}}{1 - e^{2\pi(\beta - \alpha + l/2)}} \\
&= -i \left(1 + 2 \sum_{n=1}^{\infty} e^{2\pi i n(\beta - \alpha + l/2)} \right)
\end{aligned} \tag{43}$$

where in the second identity, we have employed the formal expansion that was used in

[9]. With this R becomes,

$$\begin{aligned}
R = & \left[e^{-i\pi\epsilon} - 2\pi i \left(\frac{Q_1 Q_5 (w^2 - \lambda^2)}{4R^4} \right)^{l+1} \gamma^{2(l+1)} \frac{\xi_0}{[l!(l+1)!]^2} \right. \\
& \times \left(\frac{\Gamma'(1 + \alpha + \beta + \frac{l}{2})}{\Gamma(1 + \alpha + \beta + \frac{l}{2})} + \frac{\Gamma'(1 + \beta - \alpha + \frac{l}{2})}{\Gamma(1 + \beta - \alpha + \frac{l}{2})} - i\pi - \sum_{n=1}^l \frac{1}{n} + 2\gamma_0 \right) \left. \right] \\
& - 4\pi^2 \left(\frac{Q_1 Q_5 (w^2 - \lambda^2)}{4R^4} \right)^{l+1} \gamma^{2(l+1)} \frac{\xi_0}{[l!(l+1)!]^2} \sum_{n=1}^{\infty} e^{2\pi i n(\beta - \alpha + l/2)} \quad (44)
\end{aligned}$$

From the last term which contains the exponential factor, one can read off the time delay between two adjacent quanta that emerge from the tube,

$$\Delta t = 2\pi \frac{d}{d\tilde{\omega}}(\beta - \alpha) = \pi \frac{R}{\gamma} = \pi \frac{\sqrt{Q_1 Q_5}}{a}. \quad (45)$$

This is precisely the same as the result in [9]: neither the momentum on the compact direction nor the generic value of γ affects the time delay. However, both of them do affect the probability for a quantum to go into the throat. The probability is given as a square of the coefficient of the exponential factor,

$$\begin{aligned}
P &= \left[4\pi^2 \left(\frac{Q_1 Q_5 (w^2 - \lambda^2)}{4R^4} \right)^{l+1} \frac{\gamma^{2(l+1)}}{[l!(l+1)!]^2} \xi_0 \right]^2 \quad (46) \\
&= \left[4\pi^2 \left(\frac{Q_1 Q_5 (w^2 - \lambda^2)}{4R^4} \right)^{l+1} \frac{\gamma^{2(l+1)}}{[l!(l+1)!]^2} \frac{\Gamma(1 + \alpha + \beta + \frac{l}{2})}{\Gamma(\alpha + \beta - \frac{l}{2})} \frac{\Gamma(1 + \beta - \alpha + \frac{l}{2})}{\Gamma(\beta - \alpha - \frac{l}{2})} \right]^2
\end{aligned}$$

where in the second line we have used the first equation of (40). This reduces, as one can easily verify, to the result of [4, 9] when $\lambda = 0$ and $\gamma \rightarrow 0$, but differs from it generically.

In this note, we have considered the geometry given by (1) and a quantum placed inside. We have computed two quantities: the probability for a quantum to enter the throat, (45), and the time interval for it to round-trip the throat once, (46). We conclude with a few remarks. The geometry of (1) is such that in the throat region it is AdS whereas it becomes flat asymptotically. The dual CFT, which is an orbifold CFT, describes the AdS region. As shown above, the time delay turns out to be the same as the result of orbifold CFT that was obtained through an ‘effective string’ picture [6].

This strengthens the validity of the use of the orbifold CFT for the physics inside the throat region. Meanwhile, the flat region plays a role, as one might have expected, for the probability. In the CFT analysis, the flat region is accounted by terms that break conformal symmetry. In the leading order the ‘non-CFT’ terms give results that agree with the gravity calculation. Since we now have more general results in the gravity side, it should be possible to match them with CFT computation by including more detailed and higher order terms that arise from expanding a DBI action. Our results should be useful for that purpose.

Acknowledgements

I thank S. D. Mathur for suggesting this problem. I am also very grateful to O. Lunin for valuable discussions. This work is in part supported by DOE grant DE-FG02-91ER-40690.

Appendices

A Useful Formulae

Define $\xi \equiv \xi_0 + \epsilon' \xi_1$. In the first order in ϵ' ,

$$\begin{aligned}
\xi &= \frac{\Gamma(q+l+1)\Gamma(q-2\alpha+l+1)}{\Gamma(q)\Gamma(q-2\alpha)} \\
&= \frac{\Gamma(1+\alpha+\beta+\frac{l}{2}-\frac{\epsilon'}{2})\Gamma(1+\beta-\alpha+\frac{l}{2}-\frac{\epsilon'}{2})}{\Gamma(\alpha+\beta-\frac{l}{2}-\frac{\epsilon'}{2})\Gamma(\beta-\alpha-\frac{l}{2}-\frac{\epsilon'}{2})} \\
&\simeq \frac{\Gamma(1+\alpha+\beta+\frac{l}{2})-\frac{\epsilon'}{2}\Gamma'(1+\alpha+\beta+\frac{l}{2})}{\Gamma(\alpha+\beta-\frac{l}{2})-\frac{\epsilon'}{2}\Gamma'(\alpha+\beta-\frac{l}{2})} \\
&\quad \times \frac{\Gamma(1+\beta-\alpha+\frac{l}{2})-\frac{\epsilon'}{2}\Gamma'(1+\beta-\alpha+\frac{l}{2})}{\Gamma(\beta-\alpha-\frac{l}{2})-\frac{\epsilon'}{2}\Gamma'(\beta-\alpha-\frac{l}{2})} \\
&\simeq \xi_0 \left(1 + \frac{\epsilon'}{2} \left(-\frac{\Gamma'(1+\alpha+\beta+\frac{l}{2})}{\Gamma(1+\alpha+\beta+\frac{l}{2})} - \frac{\Gamma'(1+\beta-\alpha+\frac{l}{2})}{\Gamma(1+\beta-\alpha+\frac{l}{2})} \right. \right. \\
&\quad \left. \left. + \frac{\Gamma'(\alpha+\beta-\frac{l}{2})}{\Gamma(\alpha+\beta-\frac{l}{2})} + \frac{\Gamma'(\beta-\alpha-\frac{l}{2})}{\Gamma(\beta-\alpha-\frac{l}{2})} \right) \right) \tag{A.1}
\end{aligned}$$

where the prime on Γ 's represents a derivative with respect to the argument. From this one reads off

$$\begin{aligned}
\xi_0 &= \frac{\Gamma(1+\alpha+\beta+\frac{l}{2})}{\Gamma(\alpha+\beta-\frac{l}{2})} \frac{\Gamma(1+\beta-\alpha+\frac{l}{2})}{\Gamma(\beta-\alpha-\frac{l}{2})} \\
\frac{\xi_1}{\xi_0} &= -\frac{1}{2} \frac{\Gamma'(1+\alpha+\beta+\frac{l}{2})}{\Gamma(1+\alpha+\beta+\frac{l}{2})} - \frac{1}{2} \frac{\Gamma'(1+\beta-\alpha+\frac{l}{2})}{\Gamma(1+\beta-\alpha+\frac{l}{2})} \\
&\quad + \frac{1}{2} \frac{\Gamma'(\alpha+\beta-\frac{l}{2})}{\Gamma(\alpha+\beta-\frac{l}{2})} + \frac{1}{2} \frac{\Gamma'(\beta-\alpha-\frac{l}{2})}{\Gamma(\beta-\alpha-\frac{l}{2})} \tag{A.2}
\end{aligned}$$

The $\frac{\Gamma\Gamma}{\Gamma\Gamma}$ -term that appears e.g., in (34) can be approximated as follows,

$$\begin{aligned}
& \frac{\Gamma(\frac{1}{2} + \alpha + \beta + \frac{\nu'}{2})\Gamma(\frac{1}{2} + \alpha - \beta + \frac{\nu'}{2})}{\Gamma(\frac{1}{2} + \alpha + \beta - \frac{\nu'}{2})\Gamma(\frac{1}{2} + \alpha - \beta - \frac{\nu'}{2})} \\
&= \frac{\Gamma(\frac{1}{2} + \alpha + \beta + \frac{\nu'}{2})\Gamma(\frac{1}{2} + \beta - \alpha + \frac{\nu'}{2})}{\Gamma(\frac{1}{2} + \alpha + \beta - \frac{\nu'}{2})\Gamma(\frac{1}{2} + \beta - \alpha - \frac{\nu'}{2})} \frac{\sin \pi(\alpha - \beta + \frac{1-\nu'}{2})}{\sin \pi(\alpha - \beta + \frac{1+\nu'}{2})} \\
&\simeq \frac{\Gamma(1 + \alpha + \beta + \frac{l}{2} + \frac{\epsilon'}{2})\Gamma(1 + \beta - \alpha + \frac{l}{2} + \frac{\epsilon'}{2})}{\Gamma(\alpha + \beta - \frac{l}{2} - \frac{\epsilon'}{2})\Gamma(\beta - \alpha - \frac{l}{2} - \frac{\epsilon'}{2})} \\
&\quad \left(1 + \pi\epsilon' \frac{\cos \pi(\beta - \alpha + \frac{l}{2})}{\sin \pi(\beta - \alpha + \frac{l}{2})}\right) (-1)^{l+1}
\end{aligned} \tag{A.3}$$

where in the first equality we have used

$$\Gamma(x) = \frac{\pi}{\sin \pi x} \frac{1}{\Gamma(1-x)}, \tag{A.4}$$

and in the second equality a trigonometric identity. In the first order of ϵ' the four Γ -piece that appears in the third line of (A.3) can be written as

$$\begin{aligned}
& \frac{\Gamma(1 + \alpha + \beta + \frac{l}{2} + \frac{\epsilon'}{2})\Gamma(1 + \beta - \alpha + \frac{l}{2} + \frac{\epsilon'}{2})}{\Gamma(\alpha + \beta - \frac{l}{2} - \frac{\epsilon'}{2})\Gamma(\beta - \alpha - \frac{l}{2} - \frac{\epsilon'}{2})} \\
&\simeq \frac{(\Gamma(1 + \alpha + \beta + \frac{l}{2}) + \frac{\epsilon'}{2}\Gamma'(1 + \alpha + \beta + \frac{l}{2}))}{(\Gamma(\alpha + \beta - \frac{l}{2}) - \frac{\epsilon'}{2}\Gamma'(\alpha + \beta - \frac{l}{2}))} \\
&\quad \times \frac{(\Gamma(1 + \beta - \alpha + \frac{l}{2}) + \frac{\epsilon'}{2}\Gamma'(1 + \beta - \alpha + \frac{l}{2}))}{(\Gamma(\beta - \alpha - \frac{l}{2}) - \frac{\epsilon'}{2}\Gamma'(\beta - \alpha - \frac{l}{2}))} \\
&\simeq \frac{\Gamma(1 + \alpha + \beta + \frac{l}{2})\Gamma(1 + \beta - \alpha + \frac{l}{2})}{\Gamma(\alpha + \beta - \frac{l}{2})\Gamma(\beta - \alpha - \frac{l}{2})} \left(1 + \frac{\epsilon'}{2} \frac{\Gamma'(1 + \alpha + \beta + \frac{l}{2})}{\Gamma(1 + \alpha + \beta + \frac{l}{2})}\right) \\
&\quad \left(1 + \frac{\epsilon'}{2} \frac{\Gamma'(1 + \beta - \alpha + \frac{l}{2})}{\Gamma(1 + \beta - \alpha + \frac{l}{2})}\right) \left(1 + \frac{\epsilon'}{2} \frac{\Gamma'(\alpha + \beta - \frac{l}{2})}{\Gamma(\alpha + \beta - \frac{l}{2})}\right) \left(1 + \frac{\epsilon'}{2} \frac{\Gamma'(\beta - \alpha - \frac{l}{2})}{\Gamma(\beta - \alpha - \frac{l}{2})}\right) \\
&\simeq \xi_0 \left(1 + \frac{\epsilon'}{2} \left(\frac{\Gamma'(1 + \alpha + \beta + \frac{l}{2})}{\Gamma(1 + \alpha + \beta + \frac{l}{2})} + \frac{\Gamma'(1 + \beta - \alpha + \frac{l}{2})}{\Gamma(1 + \beta - \alpha + \frac{l}{2})}\right.\right. \\
&\quad \left.\left.+ \frac{\Gamma'(\alpha + \beta - \frac{l}{2})}{\Gamma(\alpha + \beta - \frac{l}{2})} + \frac{\Gamma'(\beta - \alpha - \frac{l}{2})}{\Gamma(\beta - \alpha - \frac{l}{2})}\right)\right)
\end{aligned} \tag{A.5}$$

Therefore

$$\begin{aligned}
& \frac{\Gamma(\frac{1}{2} + \alpha + \beta + \frac{\nu'}{2})\Gamma(\frac{1}{2} + \alpha - \beta + \frac{\nu'}{2})}{\Gamma(\frac{1}{2} + \alpha + \beta - \frac{\nu'}{2})\Gamma(\frac{1}{2} + \alpha - \beta - \frac{\nu'}{2})} \\
\simeq & \xi_0 \left[1 + \frac{\epsilon'}{2} \left(\frac{\Gamma'(1 + \alpha + \beta + \frac{l}{2})}{\Gamma(1 + \alpha + \beta + \frac{l}{2})} + \frac{\Gamma'(1 + \beta - \alpha + \frac{l}{2})}{\Gamma(1 + \beta - \alpha + \frac{l}{2})} \right. \right. \\
& \left. \left. + \frac{\Gamma'(\alpha + \beta - \frac{l}{2})}{\Gamma(\alpha + \beta - \frac{l}{2})} + \frac{\Gamma'(\beta - \alpha - \frac{l}{2})}{\Gamma(\beta - \alpha - \frac{l}{2})} \right) \right] \left(1 + \pi \epsilon' \frac{\cos \pi(\beta - \alpha + \frac{l}{2})}{\sin \pi(\beta - \alpha + \frac{l}{2})} \right) (-1)^{l+1}
\end{aligned} \tag{A.6}$$

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